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COHOMOLOGY OF LINE BUNDLES ON G/B FOR THE EXCEPTIONAL GROUP G_2

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1. Introduction

The cohomology of line bundles on G/B, where G is a semisimple algebraic group and B a Borel subgroup, is gradually becoming understood in prime characteristic. In particular, the work of Andersen (cf. [1–6]) has revealed much about the vanishing and nonvanishing of this cohomology, as well as aspects of the G-module structure. But there appear to be substantial obstacles to getting a complete picture, in part because of unproved conjectures about the relationship of Andersen's filtrations [5] to the inverse Kazhdan–Lusztig polynomials for the affine Weyl group associated with G, cf. [9]. Our aim here is to report on the current state of knowledge about the rank-2 group G_2, based on extensive empirical study using Andersen's methods. This may help to clarify what still needs to be done and to strengthen some of the conjectures in [9].

2. Notation

We will need some general notation, for which we follow mainly the conventions of Andersen [1–6], cf. also [8–9].

K	algebraically closed field of characteristic $p>0$,
G	simply connected semisimple algebraic group over K,
B	Borel subgroup of G corresponding to negative roots,
T	maximal torus of G contained in B,
$X(T)$	character group of T (full weight lattice of the root system),
X_{root}	root lattice in $X(T)$,
X^+	set of dominant weights in $X(T)$,
ϱ	sum of fundamental dominant weights,
N	$\dim G/B$ (= number of positive roots),

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$L(\lambda)$	irreducible G-module of highest weight $\lambda \in X^+$,
$V(\lambda)$	Weyl module of highest weight $\lambda \in X^+$,
$H^i(\lambda)$	i-th sheaf cohomology group of G/B relative to line bundle induced by $\lambda \in X(T)$,
W	Weyl group of G,
W_a	affine Weyl group (generated by W and translations by pX_{root}),
$w \cdot \lambda$	$w(\lambda + \varrho) - \varrho$ for $w \in W_a$ and $\lambda \in X(T)$,
w_0	longest element of W.

For the dot action of X_a, the origin of $X(T)$ is placed at $-\varrho$. The euclidean space $X(T) \otimes \mathbb{R}$ is partitioned into closures of alcoves, fundamental domains for the dot action of W_a. By a 'restricted weight' we mean a dominant weight whose coefficients relative to fundamental dominant weights lie between 0 and $p-1$. In rank 2 these lie in a parallelogram with lowest point $-\varrho$ and highest point $(p-1)\varrho$, contained in a union of closed alcoves (12 in the case of G_2). Following [14], we call any translate of this parallelogram a 'box'; its highest point is a special point (intersection of all possible types of affine reflecting hyperplanes).

When G is the group G_2, we denote by s_1 and s_2 respectively the reflections in W with respect to the short and long simple roots. If $p > 6$ (the Coxeter number of G_2), there are weights inside alcoves, to which most of our considerations apply equally. We number some of the dominant alcoves as in Fig. 1, for easy reference. If λ_1 denotes a typical weight inside the alcove marked 1, we write λ_2 for its image under W_a in alcove 2, etc. We say that an alcove is of 'type 1' if it is a translate of the alcove marked 1, and so on, for the various restricted alcoves 1, 2, 3, 4, 5, 6, 7, 8, 11, 13, 15, 16. To facilitate the discussion of examples, we shall often limit ourselves to weights in the 'lowest p^2-alcove' (an alcove for the affine Weyl group relative to p^2) of the dominant Weyl chamber and their images under the dot action of W. But most of the ideas carry over to the general case.

3. Generic decomposition patterns

We begin by recalling some results of Jantzen [10] on Weyl modules, limiting ourselves to weights which lie in the lowest p^2-alcove as indicated above. When a dominant weight λ lies inside an alcove sufficiently far from the walls of the dominant Weyl chamber, the pattern of composition factors of $V(\lambda)$ depends only on the type of alcove in which λ lies. The corresponding 'generic decomposition pattern' consists of the alcoves which contain linked weights $\mu = w \cdot \lambda$ ($w \in W_a$) for which $L(\mu)$ occurs as a composition factor of $V(\lambda)$; each such alcove is labelled with the multiplicity of $L(\mu)$ as a composition factor. For G_2 there are 12 alcove types, corresponding to the 12 alcoves in the restricted box; hence there are 12 generic patterns. The various patterns are obtainable from one another by a kind of rearrangement (described below). In particular, all patterns involve the same number of alcoves and the same distribution of multiplicities. For G_2 the total number of composition factors is 119. One of the 12 patterns is pictured in Fig. 2. Here the digits 0 to 6 inside alcoves refer to the Jantzen filtration levels (discussed below). These can be disregarded for the moment. Composition factor multiplicities from 1 to 4 are indicated by the presence of that many digits in an alcove. The required data to compute this pattern can be found in [11]; cf. also the 'dual' patterns in [14].

As long as λ lies far enough inside the dominant chamber to permit all alcoves in its generic pattern also to be dominant, the interpretation is clear. When this is no longer so, it is still possible to read off the composition factors of $V(\lambda)$, where λ is still assumed to lie inside an alcove. The algorithm is as follows: Consider each alcove in the pattern which lies outside the dominant chamber. Find the special

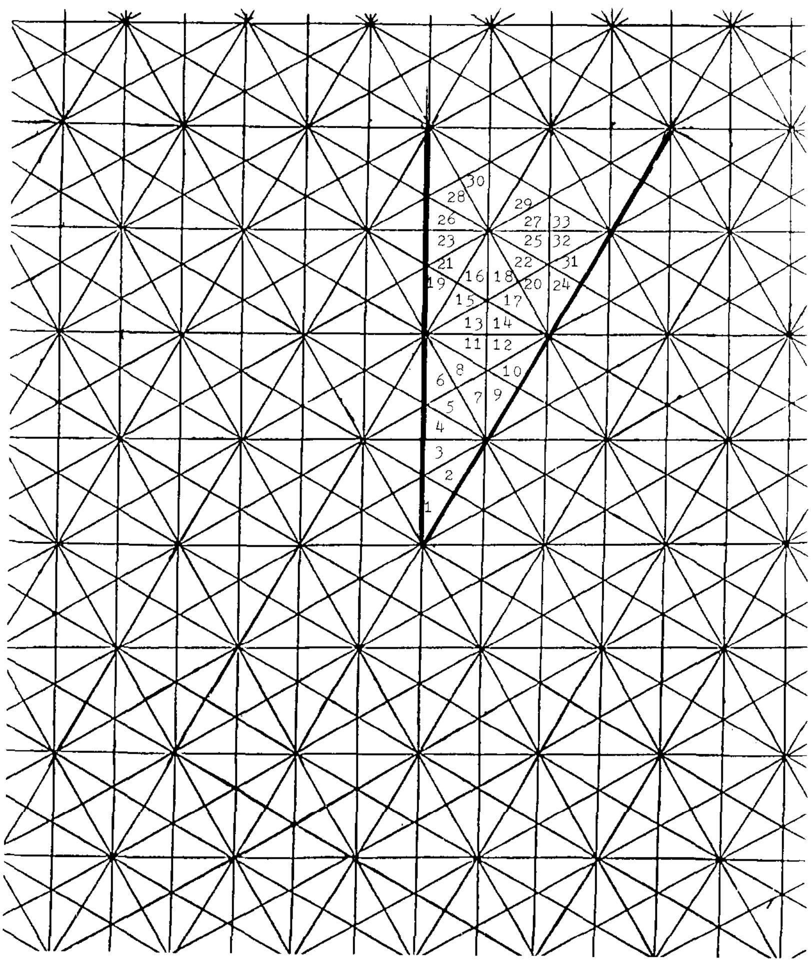

Fig. 1.

point at the top of the unique box in which that alcove lies. If that point lies on a reflecting hyperplane through $-\varrho$, discard the alcove. Otherwise there is a unique element $w \in W$ taking that point to the special point at the top of a box in the dominant chamber. Find the alcove in this box corresponding to the given alcove, and attach to it the multiplicity in the given alcove, with a sign equal to det w. After carrying out this process for all alcoves, and cancelling multiplicities if necessary, the end result is the pattern of composition factors of $V(\lambda)$.

Fig. 3 illustrates this algorithm when λ is in alcove 4. The bold lines indicate the dominant chamber. The alcoves marked 2 and 3 at the right correspond to the dominant alcoves 1 and 2 which are also marked 2 and 3 respectively. A single reflection from W is involved here, so there is a cancellation of alcoves 1 and 2 from the picture. All other alcoves to the right of the s_2-wall disappear, since the special points

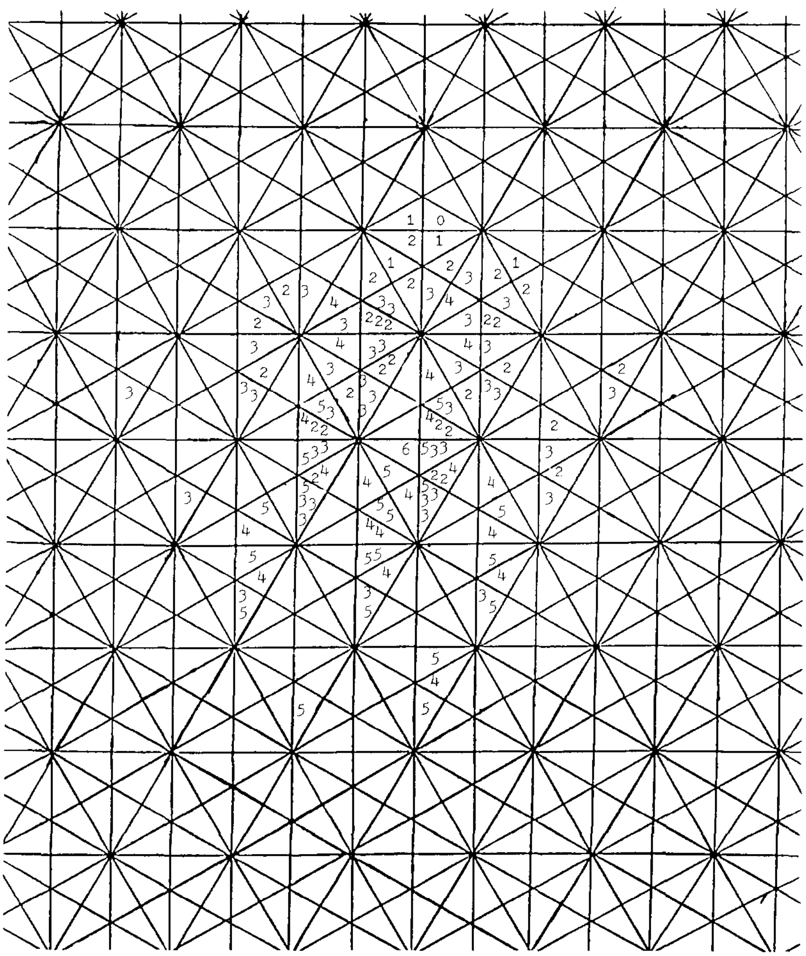

Fig. 2.

at the tops of their boxes lie in reflecting hyperplanes, through $-\varrho$. To the left of the s_1-wall things are a bit more complicated. The alcove marked 3 at the far left is in a box whose top point requires two reflections to get into the dominant chamber, while the other outlined alcove marked 3 involves just one reflection. The two signs accordingly cancel each other. The other alcoves to the left of the s_1-wall disappear. So the net result is that $V(\lambda)$ will have only two composition factors, corresponding to λ in alcove 4 and the reflected weight in alcove 3. While we have not yet explained the role of the numbers in the alcoves here, it should be observed that the numbers match neatly for alcoves involved in this cancellation process.

For a weight λ not lying inside an alcove, the composition factors of $V(\lambda)$ are obtained by using Jantzen's translation principle (cf. [4]). Find the alcove in whose 'upper closure' λ lies and compute the pattern as above for an interior weight of

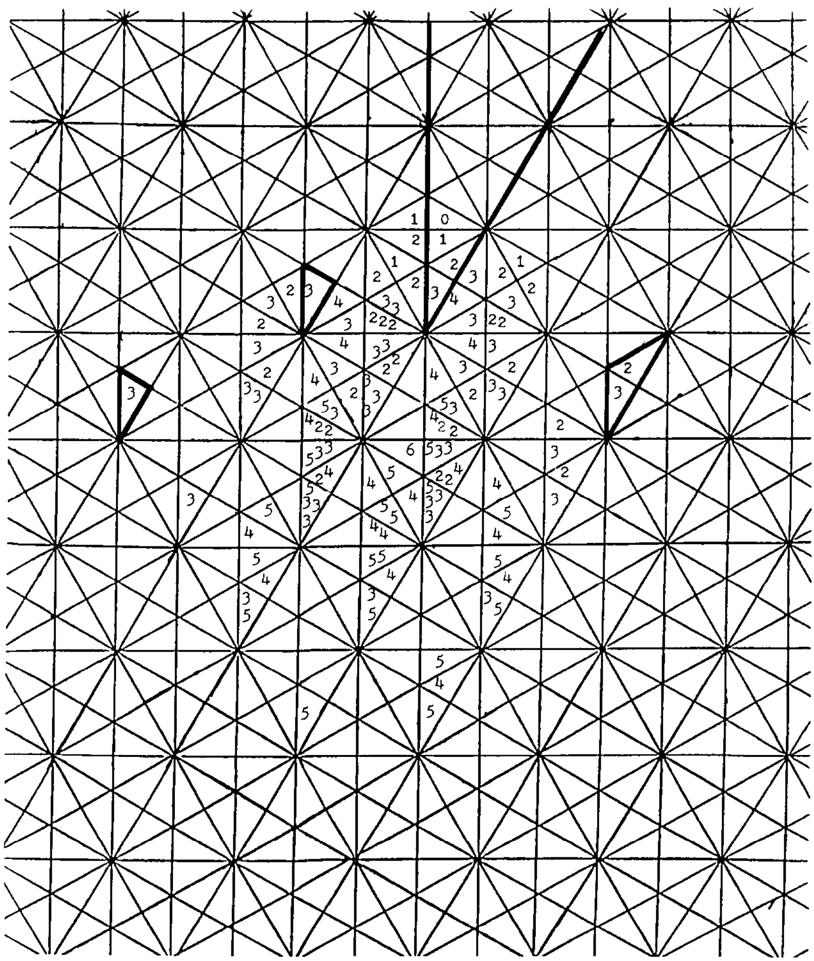

Fig. 3.

this alcove. Then translate all weights involved to the type of wall in which λ lies; only those in upper closures of alcoves survive to give composition factors of $V(\lambda)$.

4. Jantzen filtrations

For any dominant weight λ, the Weyl module $V=V(\lambda)$ has a canonical filtration $V=V^0 \supseteq V^1 \supseteq \cdots$ (cf. [12]). Here V^1 is the unique maximal submodule of V. The special feature of this Jantzen filtration is its associated 'sum formula', expressing explicitly in terms of Weyl characters the sum of formal characters of the submodules $V^1, V^2, \ldots$. When V has no multiple composition factors, this formula determines (in principle) the precise layer V^i/V^{i+1} in which each composition factor lies. When (as in the case of G_2) there are multiplicities to contend with, this is no longer so. However, in this special case Jantzen has been able to conjecture the layers by using various ad hoc arguments. We have taken his (unpublished) calculations as our starting point here. Fig. 2 indicates the layers 0 through 6 in his filtration for a generically placed weight in an alcove of type 4 in the lowest p^2-alcove. In all cases, $L(\lambda)$ alone lies in the highest layer, while in our generic situation the simple socle of $V(\lambda)$ occurs alone in layer $N=6$.

It is expected that the Jantzen filtration will turn out to be (generically) the same as the socle series, but this remains conjectural. It is also expected that the filtration layers will be correlated in a simple way with the inverse Kazhdan–Lusztig polynomials for the affine Weyl group [14], cf. the conjectures in [9]. This viewpoint has in fact been used to doublecheck many of Jantzen's computations for G_2.

5. Andersen filtrations

Andersen [5] has introduced filtrations in cohomology modules, with associated sum formulas, along the lines conjectured in [9]. In the top degree N, he recovers the Jantzen filtration of $V(\lambda)=H^N(w_0\cdot\lambda)$. Fig. 4 shows the filtration layers for a typical $H^3(s_2s_1s_2\cdot\lambda)$ when λ is a generically placed dominant weight in an alcove of type 4. The distribution of composition factors agrees with Fig. 2, but the layers in which they occur are usually different.

As in Jantzen's case, the sum formula alone is inadequate to predict the layers when multiple composition factors occur. But Fig. 4 does agree with the conjecture in [9] concerning the connection with Kazhdan–Lusztig polynomials. Perhaps some ad hoc arguments would settle matters in the case of G_2, but for our present purpose we will postulate the correctness of Fig. 4 and see what follows.

For generic weights in the lowest p^2-alcove, there are 144 patterns of the type shown in Fig. 2 and 4, since there are 12 alcove types in each of the 12 Weyl chambers. Thanks to [5, Proposition 4.6(i)], these occur in pairs, with level i replaced by level $6-i$ to get from one to the other. Thus it is enough to compute 72 of the patterns, which we have done. In practice, we have adopted a shortcut originally found in 1981 but only recently made rigorous by Kaneda [13] (in response to the rediscovery of this method by Doty and Sullivan [7]). The idea as formulated in [7] is to start with Jantzen's filtration of an H^6, then to cut the picture into strips parallel to one of the walls of the dominant chamber, and finally to reassemble the strips in the opposite order (say right to left rather than left to right). The result is another generic pattern, with an H^5 filtration. The process continues step by step for G_2 by alternating the choice of walls. Kaneda has verified in general that such a procedure converts the Jantzen sum formula into the appropriate Andersen sum formula; but again the detailed layer information for repeated composition factors must be regarded at present as conjectural.

One other feature of Fig. 4 should be emphasized. There is a unique alcove labelled 0 and a unique alcove labelled 6. This reflects a general fact due to Andersen [6]

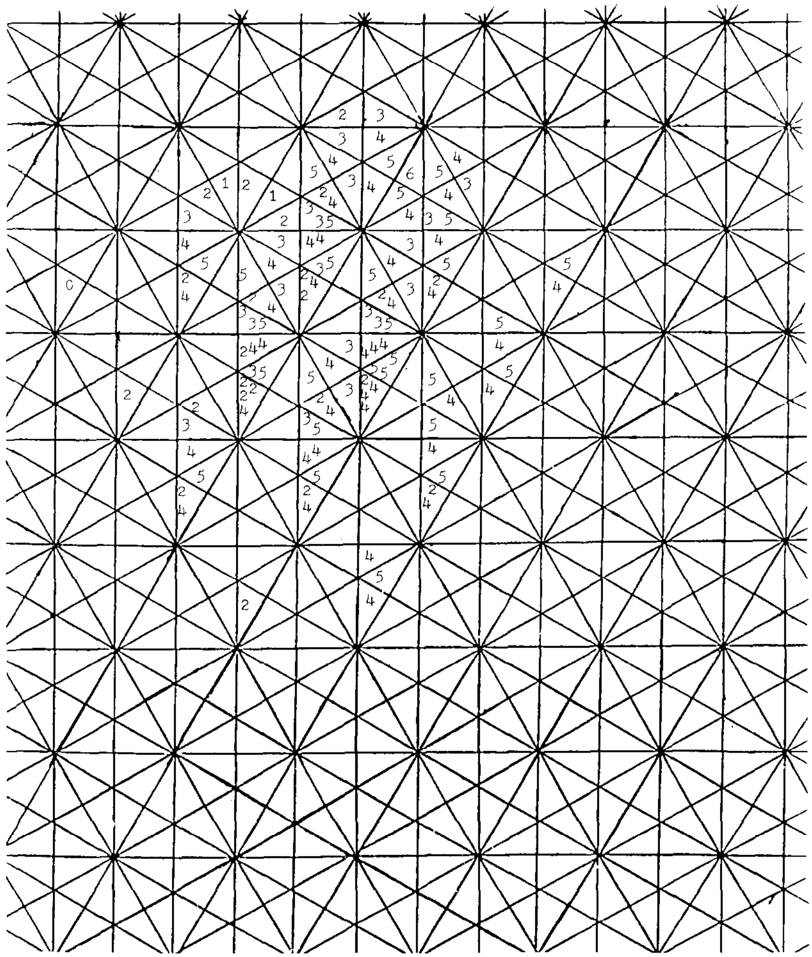

Fig. 4.

(cf. [1] for H^1): in sufficiently generic cases, cohomology modules have a simple head and simple socle (in easily predictable locations), both occurring with multiplicity 1 as composition factors. (Doty–Sullivan [7] have given another proof of this for weights in the lowest p^2-alcove.)

6. Standard and nonstandard vanishing

Having summarized the relevant facts about the cohomology modules $H^i(\lambda)$, with special reference to G_2, we can formulate our main hypothesis (which was already put forward tentatively in [9]): *The generic Andersen filtrations (conjecturally computable from the Kazhdan–Lusztig polynomials for W_a) determine the vanishing behavior and composition factors of all $H^i(\lambda)$.* To explain this in more detail (and to qualify it slightly), we begin by considering the case of 'standard vanishing': $\lambda \in X$ exhibits standard vanishing behavior if $H^i(\lambda)=0$ except for at most one value of i. If $\lambda = w \cdot \mu$ for $\mu \in X^+$, this value of i must be the length of w in W, and then $H^i(\lambda)$ has the same formal character as $H^0(\mu)$. For example, all dominant weights exhibit standard vanishing behavior, by Kempf's theorem.

In the case of G_2, Andersen displays in [4, p. 256] the vanishing pattern for weights in the lowest p^2-alcove. Roughly speaking, weights far enough from walls of Weyl chambers exhibit standard vanishing. (However, a number of the alcoves in Andersen's picture are incorrectly labelled, as will be pointed out in a couple of examples below.) Examination of a large number of filtration diagrams shows the following consistent correlation: Say λ is dominant and $w \in W$. If λ is generic as in Fig. 2, all $w \cdot \lambda$ exhibit standard vanishing behavior. If λ is degenerate as in Fig. 3, then one has to look at each $w \cdot \lambda$ separately. Say w has length i, and consider the Andersen filtation of $H^i(w \cdot \mu)$ for any generic μ in the same type of alcove as λ, as in Fig. 4. Then superimpose Weyl chamber walls as in Fig. 5 to make the highest alcove agree with the actual position of λ in the dominant chamber. Carry out the 'cancellation' algorithm of Section 3 above. If composition factors to be cancelled always lie in matching filtration layers, then $w \cdot \lambda$ turns out (in all cases computed) to exhibit standard vanishing. Otherwise it has extra nonvanishing cohomology, correlated closely with the filtration data (see below). Note for example that the cancellation in Fig. 3 goes smoothly, in conformity with Kempf's theorem, while that in Fig. 5 does not.

Although there is no clear explanation for this correlation, it points to a possible deep connection between vanishing behavior and Kazhdan–Lusztig polynomials. At any rate, it has been possible to recover in a purely formal way virtually all of Andersen's diagram in [4] (and to detect a number of errors there).

7. Long exact sequences

In order to look closely at cases of nonstandard vanishing, we need to recall the long exact sequences of Andersen [2, p. 56], cf. also [3]. The basic data consists of a weight χ, together with a simple root α, subject to the assumption that $s_\alpha \cdot \chi < \chi$. One obtains two exact sequences, which we write in an abbreviated form:

$$(1)\quad \cdots \to H^{i+1}(s_\alpha \cdot \chi) \to H^i(\chi) \to H^{i+1}(\bar{V}) \to H^{i+2}(s_\alpha \cdot \chi) \to H^{i+1}(\chi) \to H^{i+2}(\bar{V}) \to \cdots$$

$$(2)\quad \cdots \to H^i(C) \to H^i(\bar{V}) \to H^{i-1}(Q) \to H^{i+1}(C) \to H^{i+1}(\bar{V}) \to H^i(Q) \to \cdots$$

Here $\bar{V}$ is a certain B-module, while C and Q are two B-modules which have the same set of weights forming a string strictly between $s_\alpha \cdot \chi$ and χ:

$$s_\alpha \cdot \chi + p\alpha, \quad s_\alpha \cdot \chi + 2p\alpha, \ldots.$$

When at most one or two weights occur in this string, the sequences can often be used quite effectively, as we shall see below.

8. Nonvanishing in more than two degrees

It is possible in principle for $H^i(\lambda)$ to be nonzero for arbitrarily many degrees i, as Andersen has found in studying special linear groups of arbitrarily large rank. In spite of this potential complexity, our study of G_2 suggests that such nonstandard vanishing behavior may turn out to be well accounted for by the cancellation patterns in the generic Andersen filtrations when many Weyl chamber walls are crossed. We concentrate now on G_2, where we find just two distinct ways in which it is possible to get nonvanishing in more than 2 degrees.

(A) Consider first the weight $\lambda = s_2 s_1 s_2 \cdot \lambda_4$. Since λ is not dominant, both $H^0(\lambda)$ and $H^6(\lambda)$ vanish, while it follows easily from Andersen's criterion in [1] that $H^1(\lambda) = 0$. To investigate the remaining degrees, we use the long exact sequences described above, taking $\chi = \lambda$ and $s_\alpha = s_1$. There is a single intermediate weight, namely $\mu = s_2(s_1 s_2)^2 \cdot \lambda_1$. It is true in general that 'small' weights such as μ exhibit standard vanishing (e.g., apply the above exact sequences), so we have only one cohomology group $H^5(\mu) \cong L(\lambda_1)$. We claim that $s_1 \cdot \lambda$ also has standard vanishing. Applying s_2 to go down to an H^5-chamber involves no intermediate weights, so the long exact sequences allow us to do dimension shifting smoothly (as in characteristic 0). The same thing happens when we next apply s_1 to go to the H^6-chamber, where everything is standard. The upshot is that $H^4(s_1 \cdot \lambda) \cong H^6(w_0 \cdot \lambda_4)$, which has top $L(\lambda_4)$ and socle $L(\lambda_3)$; other $H^i(s_1 \cdot \lambda)$ are 0. With this data in hand, it is easy to read off the results for λ itself. Sequence (2) shows that $H^3(\bar{V}) = 0 = H^4(\bar{V})$, while $H^5(\bar{V})$ and $H^6(\bar{V})$ are isomorphic to $H^5(\mu) \cong L(\lambda_1)$. In turn, sequence (1) yields $H^2(\lambda) = 0$, $H^4(s_1 \cdot \lambda) \cong H^3(\lambda)$, $H^4(\lambda) \cong H^5(\bar{V}) \cong L(\lambda_1)$, $H^5(\lambda) \cong H^6(\bar{V}) \cong L(\lambda_1)$. (Note here that $H^3(\lambda)$ has the standard character, in spite of the extra cohomology, which just cancels out in the Euler characteristic.)

Fig. 5 shows how these results can be formally predicted from the generic Andersen H^3-filtration in Fig. 4. We interpret the Weyl chamber walls in Fig. 5 as being those of the $s_2 s_1 s_2$-chamber, so λ lies in the top alcove marked 3. The vertical wall corresponds to the wall between this chamber and the neighboring H^4-chamber. The outlined alcove marked 2 lies in a box whose top special point is in the latter chamber, and since 2 does not agree with the filtration level 4 in the congruent alcove, we expect a nonzero H^4 involving this single composition factor of type λ_1. Similarly, the special point for the alcove marked 0 lies in the H^5-chamber and should give a nonzero H^5. On the other hand, cancellation does go smoothly across the other wall, since the alcoves numbered 5 and 4 match those which they should cancel. So we expect $H^2(\lambda)=0$.

Another interesting (but atypical) phenomenon shows up when we use the technique of translation to an alcove wall, cf. [4]. Translation to the long or middle-sized wall kills $L(\lambda_1)$ by taking λ_1 into the lower closure in Jantzen's sense. In the context of the weight λ in the preceding paragraph, which lies in the fourth alcove of an H^3-chamber, these types of translation take the various H^i into their counterparts for the weight in the wall. Thus a weight in the wall separating the fourth from the fifth (resp. third) alcove has standard vanishing behavior, since its H^4 and H^5 now vanish. This possibility is overlooked in [4], but is fortunately quite rare for G_2. What happens more often is that the 'extra' H^4 and H^5 involve a number of composition factors, not all of which disappear when translated to a given wall.

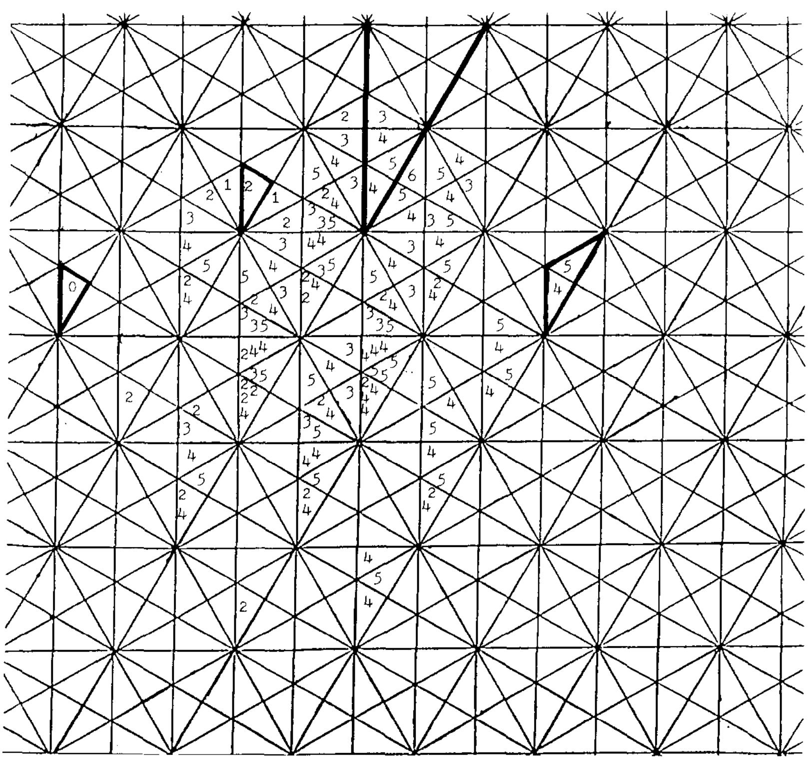

Fig. 5.

(B) The second (and somewhat more prosaic) way in which multiple nonvanishing occurs for G_2 is illustrated by Fig. 6. Here the highest weight λ in the picture lies in the fifth alcove of the same H^3-chamber as above, so again the vertical wall separates this chamber from an H^4-chamber. The outlined alcove marked 1 alone fails to cancel smoothly across this wall, so we expect to get $H^4(\lambda) \cong L(\lambda_1)$. Across the other wall we see that the alcove marked 5 alone fails to cancel smoothly, so we expect to get $H^2(\lambda) \cong L(\lambda_3)$. Use of the long exact sequences, starting with some auxiliary weights easier to analyze, confirms this prediction. Moreover, one gets along the way some insight into the precise module structure of $H^3(\lambda)$, whose composition factors are in alcoves 1, 3, 4, 5 (whereas the Euler characteristic yields just 4 and 5). For example, $L(\lambda_1)$ is actually a direct summand, since our knowledge of Weyl modules shows at once that this irreducible module cannot extend any of the

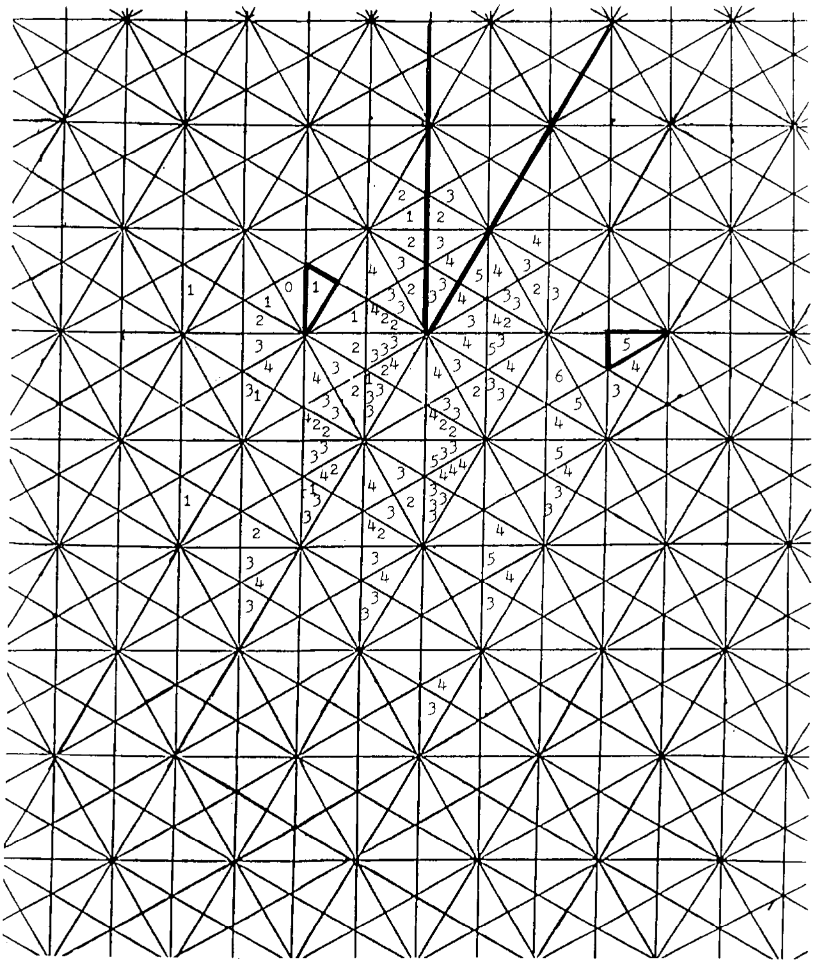

Fig. 6.

other composition factors. This decomposability is of course a highly nongeneric feature, cf. [6].

Only these two kinds of multiple nonvanishing have been observed in G_2, but some cases become much more complicated than those in Fig. 5 and 6 due to the presence of many more composition factors in the extra cohomology. In higher ranks one might of course encounter a mixture of types (A) and (B) in the same picture. One further qualification is needed, already in the case of G_2: In a situation like type (B), a multiple composition factor involving several filtration layers might allow some smooth cancellation across either (but not both) of the walls, together with some non-matching filtration numbers. Then some further investigation (using Andersen's methods) is needed before one can be certain exactly how much nonstandard vanishing occurs and in which 'extra' cohomology groups the composition factor in question occurs.

9. Semistandard vanishing

A glance at Andersen's picture in [4, p. 256] shows that the multiple nonvanishing cohomology just discussed is concentrated in a region close to $-\varrho$, where the generic patterns can overlap more than one Weyl chamber wall. As one moves further away, the only nonstandard vanishing observed occurs along a single wall; this might be dubbed 'semistandard vanishing behavior'. For example, by shifting the position of walls in Fig. 5 or 6, one soon arrives at examples of this behavior. All cases examined so far show a precise correlation between Andersen's results (sometimes corrected) and the failure of cancellation to occur smoothly in the filtration patterns. As one moves away from $-\varrho$ along a wall, the alcove marked 0 or 6 begins to play a consistent leading role in the determination of whether or not the cancellation does go smoothly. Of course, these numbers occur uniquely in the generic patterns and therefore can never cancel anything. But conversely, cancellation does seem to go smoothly (far enough away from $-\varrho$) whenever these two alcoves fail to contribute to the cancellation. This observation is purely empirical, but is suggestive of some regularity in the semistandard cases: The 'infinitesimal' methods in [6] will undoubtedly be important in working this out further in general.

10. Conclusion

What do these observations about G_2 contribute to our understanding of the general case? As Andersen's experimentation with SL_n shows, nonvanishing in many degrees can occur and will be complicated to explain in detail; but it may turn out to be limited to the region near $-\varrho$ and to be closely correlated with the cancellations in the generic decomposition patterns when numerous walls are crossed. On the other hand, there will undoubtedly turn out to be regions of semistandard vanishing which might be well explained in terms of such cancellation. Moreover, the case of G_2 provides substantial hope that the filtration levels in the generic case will predict much (if not all) of the structure of the cohomology modules in degenerate cases. One immediate goal is to reinterpret Kempf's theorem in this light, i.e., to see that cancellation for the dominant chamber always goes smoothly (as in Fig. 3 and other variants of Fig. 2).

Acknowledgements

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